Combinations of Functions

We have simple arithmetic combinations of functions inherited from arithmetic combinations of numbers as follows.

Ex 1: If $f(x) = x^2$ and $g(x) = 2x + 1$ then f + g can be thought of as a single function defined by $(f + g)(x) = f(x) + g(x) = x^2 + 2x + 1$.

Similarly we can subtract multiply and divide functions. These combinations are not very interesting.

Composition of functions is a combination that cannot be done with numbers. We define the composition of two functions f(x) and g(x) [in that order] to be f(g(x)). That is to say, we plug g(x) into f. We denote this compositions by $f\circ g(x)$. We read it as "f circle g of x".

First note that this is ***not*** multiplication. Secondly, it is not even commutative. That means $f\circ g(x)$ is not the same as $g\circ f(x)$, except in rare cases.

Ex 2: Suppose $f(x) = 2x + 5$ and $g(x) = 3x – 7$. Write expressions for

a. $f\circ g(x) =$
b. $g\circ f(x) =$

Solution:

a. First be sure that you are clear which function goes inside the other. It is a good practice to eliminate the circle notation, being careful to keep the letters in the same order. So we want f(g(x)). Build a framework for f, but parentheses for place holders instead of the variable x, which is actually just a place holder.
So we have $f\circ g(x) = f(g(x)) = 2(\quad) + 5$.
Now we drop g(x) into the space to get $f\circ g(x) = f(g(x)) = 2(3x – 7) + 5$. At this point we have no reason to multiply this out.
Notice that the factor of 2 must distribute to both the terms 3x and 7. It is very frequently necessary to include parentheses in composing functions. This is quite possibly why function notation uses parentheses.
b. $g\circ f(x) = g(f(x)) = 3(\qquad) – 7 = 3(2x + 5) – 7$.

The process of composing functions represents combining processes modeled by these functions so each process' output is fed into another process as input. These may be industrial production processes, subsidiary companies in a large corporation or parts of a multi-stage stereo amplifier.

When we analyze such systems we may decompose a function into simpler functions. Natural decompositions are usually indicated by parentheses as in Ex 2.

Ex 3: Give a decomposition of $f(x) = (3x + 7)^{17}$ as $f = g\circ h$

g(x) =
h(x) =

Solution: Again, write $g\circ h(x) = g(\mathbf{h(x)}) = \mathbf{(3x + 7)}^{17}$.

Now we have h(x), the inside part of the calculation, as 3x + 7, $g(x) = x^{17}$, the outside or final part of the calculation.

$g(x) = x^{17}$ $\qquad$ $h(x) = 3x + 7$

In Ex 3, h(x) could have been just 3x. The nest example leaves very little choice about the decomposition.

Ex 4: Give a decomposition of $h(x) = (3x + 7)^2 + 5(3x + 7) – 2$ as $h = f\circ g$

f(x) =

g(x) =

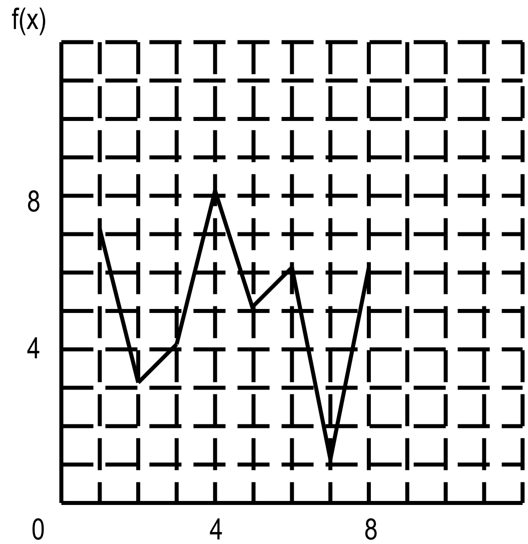
Solution: Again, write f◦g(x) = f(g(x))

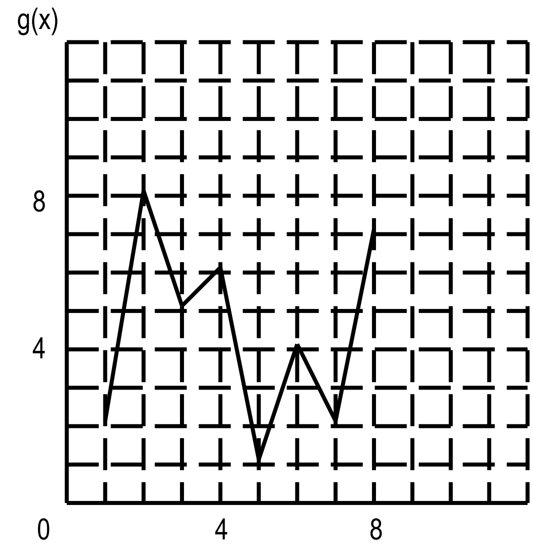
What happens to x first is that we apply the linear function 3x + 7 and then use the output twice in another function. Use the parentheses as a guide.

g(x) = 3x + 7 f(x) = x^2 + 5x – 2.

Ex 5: Evaluate these compositions using the given graphs.

f◦g(3) = f◦f(5) = g◦f(7) =





f◦g(3) = f(g(3)) = f(5) = 5

f◦f(2) =f(f(2) = f(3) = 4

g◦f(7) = g(f(7)) = g(1) = 2

Ex 6: Evaluate these compositions using the given tables.

f◦g(5) =

g◦g(2) =

g◦f(7) =

x	**f(x)**
1	2
2	5
3	3
4	4
5	7
6	3
7	5
8	4

x	**g(x)**
1	8
2	1
3	3
4	5
5	4
6	6
7	7
8	5

f◦g(5) = f(g(5)) = f(4) = 4

g◦g(2) = g(g(2) = g(1) = 2

g◦f(7) = g(f(7)) = g(5) = 4

Finally, consider a composition where y = f(x) and x = g(t). We might even write y = y(x) and x = x(t). Then y = f◦g(t) describes the relationship of y to t. We have reason to study the rate of change of y with respect to t. We use use the following analysis:

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta t}$$

This leads to a result called the Chain Rule, which is very important in calculus. Understand it as saying that the rate of change of a composition is the product of the rates of change of the separate functions. It is true even for averages rates of change of non-linear functions.

Ex 6: Suppose that f(x) = 2x – 7 and g(x) is given by the table shown. Find the average rate of change of f∘g(x) over the interval [2,12].

x	**g(x)**
2	3
4	7
6	9
8	13
10	17
12	18
14	26
16	35

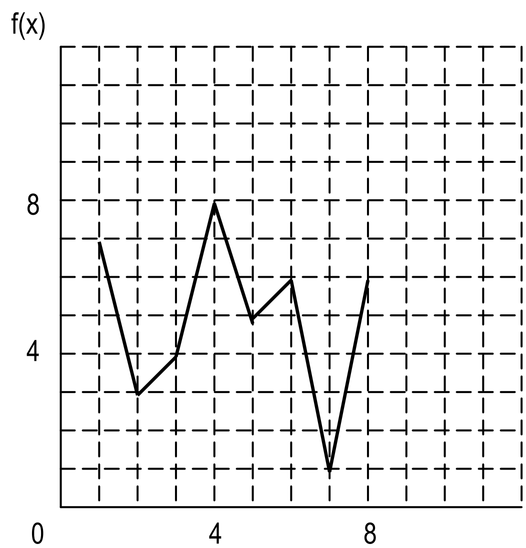
Solution:
The average rate of change of f over any interval is its slope 7.
We want to multiply that by the average rate of change of g over [2,12], which is

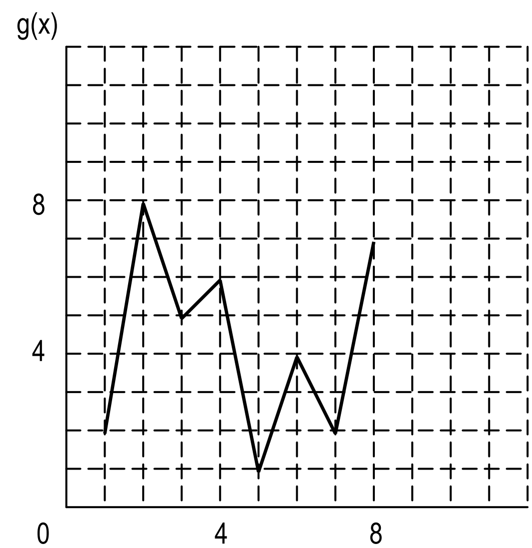
$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{18-3}{12-2} = \frac{15}{10} = 1.5$$

Twice 1.5 gives 3, so **ae** is the answer.

a. 1 **b**. 1.5 **c**. 2 **d**. 2.5
ae. 3 **be**. 3.5 **ce**. 4 **de**. 4.5

Exercises:





1. Use the graphs above to evaluate the following
i. f∘g(2) **ii**. g∘g(7) **iii**. f∘f(3) **iv**. g∘f(2)
a. 1 **b**. 2 **c**.3 **d**.4
ae. 5 **be**. 6 **ce**.7 **de**.8

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2. Use the tables at the right to evaluate the following

i. f◦g(2) **ii**. g◦g(7) **iii**. f◦f(3) **iv**. g◦f(2)
a. 1 **b**. 2 **c**.3 **d**.4
ae. 5 **be**. 6 **ce**.7 **de**.8

x	f(x)
1	2
2	5
3	3
4	4
5	7
6	3
7	5
8	4

x	g(x)
1	8
2	1
3	3
4	5
5	4
6	6
7	7
8	5

3. If f(x) = 3x + 5 and g(x) is given by the table to the right, evaluate
i. The rate of change of f◦g(x) over the interval [8,12]
ii. The rate of change of f◦g(x) over the interval [2,6]
iii. The rate of change of f◦g(x) over the interval [6,10]
iv. The rate of change of f◦g(x) over the interval [12,16]
a. 10.5 **b**. 11.25 **c**. 15 **d**. 17.5
ae. 22.5 **be**. 25 **ce**. 27.5 **de**. 30

x	g(x)
2	5
4	15
6	20
8	30
10	50
12	60
14	90
16	100

4. Suppose that f(x) = 4x – 200 and g(x) is given by the table at the right. Find the average rate of change of f◦g(x) over the intervals
i. [2,12]
ii. [6,14]
iii. [12,16]
iv. [8,16]

x	g(x)
2	92
4	87
6	80
8	73
10	61
12	58
14	45
16	38

Answers:
1.
i. f◦g(2) = f(g(2)) = f(8) = 6 **be** **ii**. g◦g(7) = g(g(7)) = g(3) = 5 **ae**
iii. f◦f(3) = f(f(3)) = f(4) = 8 **de** **iv**. g◦f(2) = g(f(2)) = g(3) = 5 **ae**
2.
i. f◦g(2) = f(g(2)) = f(1) = 2 **b** **ii**. g◦g(7) = g(g(7)) = g(7) = 7 **ce**
iii. f◦f(3) = f(f(3)) = f(3) = 3 **c** **iv**. g◦f(2) = g(f(2)) = g(5) = 4 **d**

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3.

i. $3 \cdot \frac{60-30}{12-8} = \frac{60}{4} = 15$ **c**

ii. $3 \cdot \frac{15}{4} = 11.25$ **b**

iii. $3 \cdot \frac{30}{4} = 22.5$ **ae**

iv. $3 \cdot \frac{40}{4} = 30$ **de**

4.

i. $\frac{y_2 - y_1}{x_2 - x_1} = \frac{58-92}{12-2} = \frac{-34}{10} = -3.4$ 4(-3.4) = -13.6

ii. $\frac{45-80}{14-6} = \frac{-35}{8}$ $4\left(\frac{-35}{8}\right) = \frac{-35}{2}$

iii. $\frac{38-58}{16-12} = \frac{-20}{4} = -5$ 4(-5) = -20

iv. $\frac{38-73}{16-8} = \frac{-35}{8}$ $4\left(\frac{-35}{8}\right) = \frac{-35}{2}$