Determinants and Cramer's Rule

1. Find the determinant of $A = \begin{bmatrix} 2 & x \\ 3 & 4 \end{bmatrix}$

2. Find the determinant of $A = \begin{bmatrix} x & -2 \\ 2 & 7 \end{bmatrix}$

3. Find the determinant of $A = \begin{bmatrix} 2 & x \\ -3 & -4 \end{bmatrix}$

4. Find the determinant of $A = \begin{bmatrix} 2 & x \\ -7 & x \end{bmatrix}$

5. Calculate $\det\begin{bmatrix} 1 & 2 & a \\ 3 & -2 & 2 \\ 3 & 2 & -1 \end{bmatrix}$

6. Calculate $\det\begin{bmatrix} 1 & 2 & -1 \\ x & -2 & 2 \\ 3 & 2 & -1 \end{bmatrix}$

7. Calculate $\det\begin{bmatrix} 1-x & 3 \\ 5 & 4-x \end{bmatrix}$

8. Calculate $\det\begin{bmatrix} 2-x & 2 \\ 3 & 5-x \end{bmatrix}$

9. Calculate $\det\begin{bmatrix} 1-x & 2 & -1 \\ 3 & 2-x & 2 \\ 3 & 2 & 3-x \end{bmatrix}$

10. Solve $\begin{cases} x + 2y - z = 3 \\ 2x - 3y + 2z = 1 \\ -x + 2y - 2z = 2 \end{cases}$ for x using **Cramer's Rule**

Answers:

1. 8 – 3x

2. 7x + 4

3. -8 + 3x

4. 9x

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5. $\det\begin{bmatrix}1 & 2 & a\\ 3 & -2 & 2\\ 3 & 2 & -1\end{bmatrix}\begin{matrix}1 & 2\\ 3 & -2\\ 3 & 2\end{matrix}$

= 1(-2)(-1) + 2*2*3 + a*3*2 – 3(-2)*a – 2*2*1 – (-1)*3*2

= 16+ 12a

6. $\det\begin{bmatrix}1 & 2 & -1\\ x & -2 & 2\\ 3 & 2 & -1\end{bmatrix}\begin{matrix}1 & 2\\ x & -2\\ 3 & 2\end{matrix}$

= 1(-2)(-1) + 2*2*3 + (-1)*x*2 – 3*(-2)(-1) – 2*2*1 – (-1)*x*2

= 4

7. (1 – x)(4 – x) – 5·3

= x^2 – 5x + 4 – 15

= x^2 – 5x – 11

8. (2 – x)(5 – x) – 2·3

= x^2 – 7x + 10 – 6

= x^2 – 7x + 4

9. $\det\begin{bmatrix}1-x & 2 & -1\\ 3 & 2-x & 2\\ 3 & 2 & 3-x\end{bmatrix}\begin{matrix}1-x & 2\\ 3 & 2-x\\ 3 & 2\end{matrix}$

= (1 – x)(2 – x)(3 – x) + 2·2·3 + (-1)·3·2 – [3·(2 – x)·(-1) + 2·2·(1 – x) + (3 – x)·3·2]

= -(x^2 – 3x + 2)(x – 3) + 12 – 6 – [-6 + 3x + 4 – 4x + 18 – 6x]

= (-x^3 + 6x^2 – 11x + 6) – 10 + 7x

= -x^3 + 6x^2 – 4x – 4

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10. Here is the procedure for applying the theorem known as **Cramer's Rule**.

We are solving **AX = C** where $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 2 & -2 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Replacing column 1 of **A** with **B** we get a matrix whose determinant is the numerator of x. The denominator of x is det(**A**).

$$\text{So } x = \frac{\det\begin{bmatrix} 3 & 2 & -1 \\ 1 & -3 & 2 \\ 2 & 2 & -2 \end{bmatrix}}{\det\begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 2 & -2 \end{bmatrix}} = \frac{10}{5} = 2$$

Similarly, if we replace the second or third column of **A** with **C**, for the determinant in the numerator, we will get the values for y and z.

Of course this method requires that det(**A**), in the denominator, be non-zero, so $\mathbf{A}^{-1}$ must exist. Since we have to enter **A** into a calculator for a numerical problem, calculating $\mathbf{A}^{-1}\mathbf{B}$ is as easy.

Our interest in this method is partly historical, since hand calculations for a system with integer coefficients could be done without introducing fractions until the final quotient formation. More importantly, we often have interest in matrices with polynomial entries, which would require a more sophisticated computation utility.