Sequences and Series

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8.1 Sequences, Series and Σ-notation

1. Find n^{th} term of the sequence 2, 4, 6, 8, …

Note that we start this sequence with n = 1. So $a_1 = 2$.

2. Find n^{th} term of the sequence 3, 6, 9, 12, …

Note that we start this sequence with n = 1. So $a_1 = 3$

3. Find n^{th} term of the sequence 7, 11, 15, 19, 23, …

Note that we start this sequence with n = 1. So $a_1 = 7$

4. Find n^{th} term of the sequence 2, 4, 8, 16, …

Note that we start this sequence with n = 1. So $a_1 = 2$

5. Find n^{th} term of the sequence 10, 100, 1000, 10000 …

Note that we start this sequence with n = 1. So $a_1 = 10$

6. Find n^{th} term of the sequence $\frac{4}{2}, \frac{7}{4}, \frac{10}{6}, \frac{13}{8}, \cdots$

Note that we start this sequence with n = 1. So $a_1 = \frac{4}{2}$.

7. Evaluate $\sum_{n=1}^{5} \left(n^2\right)$

8. Evaluate $\sum_{n=4}^{6} \left(n^3\right)$

9. Evaluate $\sum_{n=1}^{5} (-1)^{n+1} \frac{1}{2^n}$

10. Evaluate $\sum_{n=4}^{6} (3n-5)$

8.2 Arithmetic Series

1. Find 20^{th} sum of 11 + 15 + 19 + 23 + …Note that $a_1 = 11$, and the first "sum" is 11, just the first term of the associated sequence. You will use the fact that for an arithmetic series, $\sum_{i=1}^{n} a_i = \frac{n(a_1 + a_n)}{2}$

8.3 Geometric Series

1. Find n^{th} term of $100(1.03)$, $100(1.03)^2$, $100(1.03)^3$, $100(1.03)^4$, …

Note that $a_1 = 100(1.03)$

2. Find n^{th} sum of $100(1.03) + 100(1.03)^2 + 100(1.03)^3 + 100(1.03)^4 + \ldots$Note that the first "sum" is 100(1.03), just the first term of the associated sequence.

3. Find sum of all the infinitely many terms of

$100(1.03) + 100(1.03)^2 + 100(1.03)^3 + 100(1.03)^4 + \ldots$

8.4 Mathematical induction

1. Use Mathematical Induction to prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

2. Use Mathematical Induction to prove $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

3. Use Mathematical Induction to prove $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

8.7 The Binomial Theorem is another result proven by Mathematical Induction. Our application of it in Calculus is in analyzing parts of $(x + h)^n$. This theorem states that

$$(x+h)^n = C_0^n x^n h^0 + C_1^n x^{n-1} h^1 + C_2^n x^{n-2} h^2 + \cdots C_{n-2}^n x^2 h^{n-2} + C_{n-1}^n x^1 h^{n-1} + C_n^n x^0 h^n$$

The terms have variable factors $x^{n-k} \cdot h^k$ so the exponents add up to n. The coefficient C_k^n has a constant superscript and a subscript matching the h coefficient.

C_k^n is a function of n and k which appears in various parts of mathematics, including probability, and has many different notations. Its value, for hand calculations, would be given as

$$C_k^n = \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{[k(k-1)(k-2)\cdots 3\cdot 2\cdot 1]\cdot[(n-k)(n-k-1)(n-k-2)\cdots 3\cdot 2\cdot 1]}$$

This is not as bad as it seems, even for hand calculations. For example,

$$C_4^{12} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot \mathbf{4}\cdot \mathbf{3}\cdot \mathbf{2}\cdot \mathbf{1}}{[\mathbf{4}\cdot \mathbf{3}\cdot \mathbf{2}\cdot \mathbf{1}][8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1]} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

. Many factors can be divided from the numerator and denominator.

To abbreviate the definition we use another function called the **factorial** function.

$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

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Now $C_k^n = \frac{n!}{k!(n-k)!}$ and $(x+h)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k}h^k = \sum_{k=0}^{n} C_k^n x^{n-k}h^k$

Here is what you need to do when you have a calculator.

Ex: Find the x^{17} term in $(x + h)^{29}$. **Ans**: ________________ $x^{17} \cdot h^{\_\_\_\_}$

The exponent of h is 29 – 17 ________________ $x^{17} \cdot h^{12}$

The coefficient is $C_{12}^{29} = 51{,}895{,}935$ 51,895,935 $x^{17} \cdot h^{12}$

Here is how you use C_k^n in most TI calculators:

1. Enter the superscript which is 29 in this example.

2. The 82 – 84 family has a [MATH] key with four menus, PRB being at the right. That is the probability menu. Navigate to it with the arrow keys and select the 3rd item, nCr.

3. Enter the subscript which is 17 in this example.

4. Your screen should read “29 nCr 17”. Press [ENTER] and you have the coefficient.

Exercises

1. Ex: Find the x^{12} term in $(x + h)^{19}$. **Ans**: ________________ $x^{12} \cdot h^{\_\_\_\_}$

2. Ex: Find the x^{14} term in $(x + h)^{21}$. **Ans**: ________________ $x^{14} \cdot h^{\_\_\_\_}$

3. Ex: Find the x^{7} term in $(x + h)^{15}$. **Ans**: ________________ $x^{7} \cdot h^{\_\_\_\_}$

4. Ex: Find the x^{9} term in $(x + h)^{14}$. **Ans**: ________________ $x^{9} \cdot h^{\_\_\_\_}$

Answers:

8.1

1. $a_n = 2n$

2. $a_n = 3n$

3. $a_n = 4n + 3$

4. $a_n = 2^n$

5. $a_n = 10^n$

6. $a_n = \frac{3n+1}{2n}$

7. $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

8. $4^3 + 5^3 + 6^3 = 405$

9. $(-1)^2 \frac{1}{2^1} + (-1)^3 \frac{1}{2^2} + (-1)^4 \frac{1}{2^3} + (-1)^5 \frac{1}{2^4} + (-1)^6 \frac{1}{2^5}$

$= + \frac{1}{2^1} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} = 0.34375$

10. $(3 \cdot 4 - 5) + (3 \cdot 5 - 5) + (3 \cdot 6 - 5) = 7 + 10 + 13 = 30$

8.4

1. Pf: n = 2: $1 + 2 = 3;\ 2(3)/2 = 3$ 1 point

Assume $\mathbf{1 + 2 + 3 + \ldots + k} = \mathbf{\frac{k(k+1)}{2}}$

Claim: $\mathbf{1 + 2 + 3 + \ldots + k} + (k + 1) = \frac{(k+1)(k+2)}{2}$ 1 point

Sufficient to prove $\mathbf{\frac{k(k+1)}{2}} + (k+1) = \frac{(k+1)(k+2)}{2}$

$\frac{k}{2} + 1 = \frac{(k+2)}{2}$ 1 point

which is clear. QED

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2. Pf: n = 2: $1^2 + 2^2 = 5; \frac{2*3*5}{6} = 5$ 1 point

Assume $\mathbf{1^2 + 2^2 + 3^2 + \ldots + k^2} = \frac{\mathbf{k(k+1)(2k+1)}}{\mathbf{6}}$ 1 point

Claim: $\mathbf{1^2 + 2^2 + 3^2 + \ldots + k^2} + (k + 1)^2 = \frac{(k+1)(k+2)[2(k+1)+1]}{6}$ 1 point

Sufficient to prove $\frac{\mathbf{k(k+1)(2k+1)}}{\mathbf{6}} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

$$\frac{k(2k+1)}{6} + (k+1) = \frac{(k+2)(2k+3)}{6}$$

$2k^2 + k + 6(k + 1) = 2k^2 + 7k + 6$ 1 point

, which is clear QED

3. Pf: n = 2: $1^3 + 2^3 = 9; \frac{2^2 * 3^2}{4} = 9$ 1 point

Assume $\mathbf{1^3 + 2^3 + 3^3 + \ldots + k^3} = \frac{\mathbf{k^2(k+1)^2}}{\mathbf{4}}$ 1 point

Claim: $\mathbf{1^3 + 2^3 + 3^3 + \ldots + k^3} + (k + 1)^3 = \frac{(k+1)^2(k+2)^2}{4}$ 1 point

Sufficient to prove $\frac{\mathbf{k^2(k+1)^2}}{\mathbf{4}} + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$

$$\frac{k^2}{4} + (k+1) = \frac{(k+2)^2}{4}$$

$k^2 + 4(k + 1) = k^2 + 4k + 4$ 1 point

, which is clear. QED

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8.7

1. Ex: Find the x^{12} term in $(x + h)^{19}$. **Ans**: $50388\ x^{12} \cdot h^{7}$

2. Ex: Find the x^{14} term in $(x + h)^{21}$. **Ans**: $116280\, x^{14} \cdot h^{7}$

3. Ex: Find the x^{7} term in $(x + h)^{15}$. **Ans**: $6435\, x^{7} \cdot h^{8}$

4. Ex: Find the x^{9} term in $(x + h)^{14}$. **Ans**: $2002\, x^{9} \cdot h^{5}$