Basic Formulas and Basic Differentiation

We have to be careful in calculus to distinguish constants from variables. In algebra a variable is essentially an unknown constant. This is not true in calculus. Generally we use the convention that letters near the beginning of the alphabet represent constants (e.g. a, b, c, d, e), while those near the end represent variables (e.g. s, t, u, v, w, x, y, z). The latter are sometimes dependent variables. Of course f, g, h, and so on tend to represent functions. In applications, P may represent profit. You are entitled to ask if a letter represents a constant, independent variable or a dependent variable (function).

Powers of x: $\frac{d}{dx}\left[x^n\right] = n \cdot x^{n-1}$ Here, n represents a constant, usually an integer, but this particular theorem applies for any particular real power n.

Examples

1. $\frac{d}{dx}\left[x^{23} = 23x^{22}\right]$

2. $\frac{d}{dx}\left[x^{2/3}\right] = \frac{2}{3} \cdot x^{-1/3}$

3. $\frac{d}{dx}\left[x^{-2.7}\right] = -2.7 \cdot x^{-3.7}$

4. $\frac{d}{dx}\left[x^{\sqrt{7}}\right] = \sqrt{7} \cdot x^{\sqrt{7}-1}$

5. $\frac{d}{dx}\left[x^{\pi}\right] = \pi \cdot x^{\pi-1}$

Exponential and logarithmic functions:

a. $\frac{d}{dx} e^x = e^x$

b. $\frac{d}{dx} a^x = a^x \cdot \ln(a)$ Note that a is any positive constant and x is the variable

c. $\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$ where x > 0.

d. $\frac{d}{dx}\left[\log_a(x)\right] = \frac{1}{\ln(a) \cdot x}$ where a, x > 0.

Derivatives of Constant Multiples of Functions: $\frac{d}{dx}\left[c \cdot f\right] = c \cdot \frac{df}{dx}$, where c is any constant

6. $\frac{d}{dx}\left[5 \cdot e^x\right] = 5e^x$

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7. $\frac{d}{dx}\left[\frac{1}{2}\ln(x)\right]=\frac{1}{2}\cdot\frac{1}{x}=\frac{1}{2x}$

8. $\frac{d}{dx}\left[7\cdot x^5\right]=7\cdot\frac{d}{dx}\left[x^5\right]=7\cdot 5x^4$

In general, you should routinely differentiate cx^n in one step, multiplying the exponent times the coefficient.

Derivatives of Sums of Functions: $\frac{d}{dx}[f+g]=\frac{df}{dx}+\frac{dg}{dx}$

More generally, differentiation is a *linear operator*, i.e. $\frac{d}{dx}[af\pm bg]=a\frac{df}{dx}\pm b\frac{dg}{dx}$ for any constants a and b and any differentiable functions f and g.

Examples

9. $\frac{d}{dx}\left[3x^7+9e^x\right]=21x^6+9e^x$

10. $\frac{d}{dx}\left[3x^7+9x^4-2x^3+5x-96\right]=\frac{d}{dx}\left[3x^7+9x^4+(-2)x^3+5x+(-96)\right]=21x^6+36x^3-6x^2+5$

So we can differentiate any polynomials. Sometimes some elementary algebra is necessary to write the polynomial in standard form.

11. For $f(x)=(2x^3+5x)(3x^2-4)$ find $f'(x)$

$$f(x)=6x^5-8x^3+15x^3-20x=6x^5+7x^3-20x \text{ so } f'(x)=30x^4+21x^2-20$$

12. Calculate $\frac{d}{dx}\left[\frac{x^5+\sqrt{x}+x^2+1}{x^3}\right]$.

Here, the essential part is to re-write the expression algebraically before doing the calculus.

$$f(x)=\frac{x^5+\sqrt{x}+x^2+1}{x^3}=\frac{x^5+x^{1/2}+x^2+1}{x^3}=\frac{x^5}{x^3}+\frac{x^{1/2}}{x^3}+\frac{x^2}{x^3}+\frac{1}{x^3}=x^2+x^{-5/2}+x^{-1}+x^{-3}$$

We will now do the differentiation, using the common notation f '(x) to denote the derivative of f(x). This is not acceptable for functions of more than one input, such as the area A(x,y) for a rectangle. Calculus of such functions is considerably more complicated, and we won't address it in this course.

$$\frac{d}{dx}\left[\frac{x^5+\sqrt{x}+x^2+1}{x^3}\right]=\frac{d}{dx}\left[x^2+x^{-5/2}+x^{-1}+x^{-3}\right]=2x-\frac{5}{2}x^{-7/2}-x^{-2}-3x^{-4}$$

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Be careful about negative exponents, because subtracting 1 from -3 gives -4, not -2. Of course be especially careful with fractions.

Higher Order Derivatives

There are significant situations from physics to finance where we have interest in the derivative of the derivative of a function! If $f(x) = 3x^4$, so $f'(x) = 12x^3$, then $f'(x)$ is a differentiable function. Its derivative is $36x^2$. It is commonly denoted $f''(x)$ or $f^{(2)}(x)$ or $\dfrac{d^2f(x)}{dx^2}$.

In certain situations in this course, you can be forced to do the algebra in **Ex 11** by being required to find and simplify the third derivative of a function.

Example

13. For $f(x) = \dfrac{3x^4 + x^3 - x^2 + 8\sqrt{x}}{x^2}$, find $f'''(x)$

$$f(x) = \frac{3x^4 + x^3 - x^2 + 8x^{1/2}}{x^2} = 3x^2 + x - 1 + 8x^{-3/2}$$

$$f'(x) = 6x + 1 + 8 \cdot \frac{-3}{2} x^{-5/2}$$

$$f''(x) = 6 + 8 \cdot \frac{3}{2} \cdot \frac{5}{2} x^{-7/2}$$

$$f'''(x) = -8 \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} x^{-9/2} = -105x^{-9/2} \text{ or } -105x^{-4.5}$$

If x(t) describes the position of a point on the x-axis at time t, then $\dfrac{dx}{dt}$ is the velocity v(t) of the point.

Velocity $v(t) = \dfrac{dx}{dt}(t) = x'(t)$ is a function of t. It typically has a derivative, and its derivative is $\dfrac{d}{dt}[v(t)] = \dfrac{d}{dt}\left[\dfrac{dx}{dt}\right]$, i.e $\dfrac{d^2x}{dt^2}(t)$ or $x''(t)$. The second derivative of position is known in physics and engineering as the acceleration a(t) of the object at the point x(t).

In a car undergoing constant acceleration, you feel a smooth push. With changing acceleration, you feel jerked around to some extent. So the derivative of acceleration, third derivative $x'''(t)$ of position, is sometimes called the jerk:

In finance, the interest in the second derivative generally has to do with finding the point of diminishing returns, which tells us where to balance the demands of our production processes.

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Exercises:

1. Find f '''(x) if $f(x)=\dfrac{3x^6+x-16\sqrt{x}}{x}$

2. Find $\dfrac{d^3f}{dx^3}$ if $f(x)=\dfrac{x^4+x^3-32\sqrt{x}}{x^3}$

3. Find y ''' if $y=\dfrac{x+7x^5-8\sqrt{x}}{x}$

4. Find f'''(x) for $\dfrac{(2\sqrt{x}+x)(\sqrt{x}-x^2)}{x}$

5. Find f'''(x) for $\dfrac{(\sqrt{x}+2)(x^2+1)}{x}$

6. Find f'''(x) for $f(x)=\dfrac{(x+3)(x-2)}{x}+\sqrt{7}\cdot e^x-\dfrac{e^2+1}{\pi}+\ln\left(e^2+3\right)$

7. Find f'''(x) for $f(x)=\left(3\cdot e^2+7\right)^8+5\cdot\pi^7+\dfrac{(x+3)(x-2)}{x}+\sqrt{7}\cdot e^x$

Answers:

1. $f(x)=\dfrac{3x^6+x-16\sqrt{x}}{x}$ = $3x^5 + 1 - 16x^{-1/2}$

$$f'(x)=15x^4+16\left(\frac{1}{2}\right)x^{-3/2}$$

$$f''(x)=60x^3-16\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2}$$

$$f'''(x)=180x^2+16\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{-7/2}$$

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2. $f(x)=\dfrac{x^4+x^3-32\sqrt{x}}{x^3}=x+1-32x^{-5/2}$

$f'(x)=1+32\left(\dfrac{5}{2}\right)x^{-7/2}$

$f''(x)=-32\left(\dfrac{5}{2}\right)\left(\dfrac{7}{2}\right)x^{-9/2}$

$f'''(x)=32\left(\dfrac{5}{2}\right)\left(\dfrac{7}{2}\right)\left(\dfrac{9}{2}\right)x^{-11/2}$

3. $y=\dfrac{x+7x^5-8\sqrt{x}}{x}=1+7x^4-8x^{-1/2}$

$y'=28x^3+\left(\dfrac{1}{2}\right)8x^{-3/2}$

$y''=84x^2-\left(\dfrac{1}{2}\right)\left(\dfrac{3}{2}\right)8x^{-5/2}$

$y'''=168x+\left(\dfrac{1}{2}\right)\left(\dfrac{3}{2}\right)\left(\dfrac{5}{2}\right)8x^{-7/2}$

4. $f(x)=\dfrac{(2\sqrt{x}+x)(\sqrt{x}-x^2)}{x}=\dfrac{2x-2x^{2.5}+x^{1.5}-x^3}{x}=2-2x^{1.5}+x^{0.5}-x^2$

$f'(x)=-3x^{0.5}+0.5x^{-0.5}-2x$

$f''(x)=-1.5x^{-0.5}-0.25x^{-1.5}-2$

$f'''(x)=0.75x^{-0.5}+0.375x^{-2.5}$

5. $f(x)=\dfrac{(\sqrt{x}+2)(x^2+1)}{x}=\dfrac{x^{2.5}+2x^2+x^{0.5}+2}{x}=x^{1.5}+2x+x^{-0.5}+2x^{-1}$

$f'(x)=1.5x^{0.5}+2-0.5x^{-1.5}-2x^{-2}$

$f''(x)=0.75x^{-0.5}+0.75x^{-2.5}+4x^{-3}$

$f'''(x)=-0.375x^{-0.5}-0.625x^{-2.5}-12x^{-4}$

6. $f(x)=\dfrac{x^2+x-6}{x}+\sqrt{7}\cdot e^x-\dfrac{e^2+1}{\pi}+\ln\left(e^2+3\right)=x+1-6x^{-1}+\sqrt{7}\cdot e^x-\dfrac{e^2+1}{\pi}+\ln\left(e^2+3\right)$

$$f'(x)=1+6x^{-2}+\sqrt{7}\cdot e^x$$

$$f''(x)=-12x^{-3}+\sqrt{7}\cdot e^x$$

$$f'''(x)=36x^{-4}+\sqrt{7}\cdot e^x$$

7. $f(x)=\left(3\cdot e^2+7\right)^8+5\cdot\pi^7+\dfrac{x^2+x-6}{x}+\sqrt{7}\cdot e^x=\left(3\cdot e^2+7\right)^8+5\cdot\pi^7+x+1-6x^{-1}$

$$f'(x)=1+6x^{-2}+\sqrt{7}\cdot e^x$$

$$f''(x)=-12x^{-3}+\sqrt{7}\cdot e^x$$

$$f'''(x)=36x^{-4}+\sqrt{7}\cdot e^x$$

Theorem: $\dfrac{d}{dx}\left[x^n\right]=n\cdot x^{n-1}$

Proof: We need to calculate the difference quotient $f'(x)=\lim\limits_{h\to 0}\dfrac{(x+h)^n-x^n}{h}$. We will need a general expansion for binomials $(x + h)^n$.

If we FOIL out n factors

$$(x + h)(x + h)(x + h)\cdots\cdots(x + h)$$

we get products like x^4h^{n-4} and x^7h^{n-7}. We need to know, for example, how many products x^4h^{n-4} we will have. The way we produce each such factor is that four of the (x + h)'s contribute an x and the other n – 4 of them contribute an h. The number of sets of 4 (x + h)'s we can choose from the n factors is called the binomial coefficient ${}_nC_4$. For a specific n, like 10, you can calculate it in the TI-83 as follows

1. type 10

2. press [MATH] and arrow right >>> to the PRB menu

3. nCr

4. type 4

5. press [ENTER] to evaluate "10 nCr 4", which is 210.

In general

$$(x + h)^n = {}_nC_n\cdot x^nh^0 + {}_nC_{n-1}\cdot x^{n-1}h^1 + {}_nC_{n-2}\cdot x^{n-2}h^2 + {}_nC_{n-3}\cdot x^{n-3}h^3 + \cdots\cdots + {}_nC_1\cdot x^1h^{n-1} + {}_nC_0\cdot x^0h^n$$

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For our purpose we only need to know general values for the first two binomial coefficients. How many ways can we choose for all the (x + h) factors to contribute an x? The answer is just 1, so ${}_nC_n = 1$. How many ways can we choose exactly one (x + h) factor to contribute an h with the other n – 1 contributing an x? The answer is exactly n, so ${}_nC_{n-1} = n$.

Now we can calculate the difference quotient

$$\frac{(x+h)^n - x^n}{h} = \frac{(x^n + nx^{n-1}h +_n C_2 x^{n-2}h^2 + \cdots_n C_{n-1}xh^{n-1} +_n C_n h^n) - x^n}{h}$$

$$= \frac{nx^{n-1}h +_n C_2 x^{n-2}h^2 + \cdots_n C_{n-1}xh^{n-1} +_n C_n h^n}{h} = nx^{n-1} +_n C_2 x^{n-2}h^1 + \cdots_n C_{n-1}xh^{n-2} +_n C_n h^{n-1}$$

Now every term but the first has an h factor, so the limit of the difference quotient as h approaches 0 is

$$\frac{d}{dx}\left[x^n\right] = n \cdot x^{n-1}.$$

This same formula also works for values of n other than natural numbers. The explanation for that must wait until later. Still we will use this formula even when the exponent n is 0, rational, negative, or even irrational.

Derivatives of Constant Multiples of Functions: $\frac{d}{dx}\left[c \cdot f\right] = c \cdot \frac{df}{dx}$, where c is any constant

Proof:

Derivatives of Sums of Functions:

Claim: $\frac{d}{dx}\left[f + g\right] = \frac{df}{dx} + \frac{dg}{dx}$

Proof: Again, we analyze the difference quotient.

$$\frac{(f+g)(x+h) - (f+g)(x)}{h} = \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

$$= \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \to \frac{df}{dx} + \frac{dg}{dx} \text{ as } h \to 0$$