Differentials and Marginal Analysis

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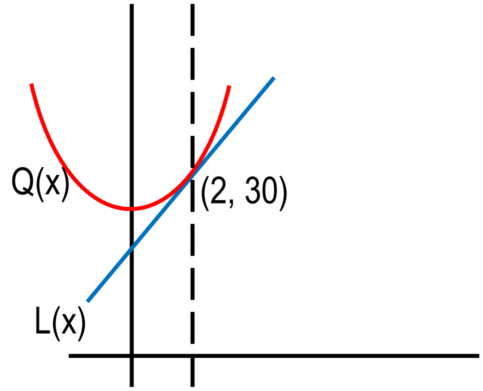
We have been studying Differential Calculus, the analysis of *small* changes in *related* variables. At the right we have a graph of $Q(x) = 5x^2 + 10$ with a good linear approximation L(x) near the value x = 2. The existence of a good linear approximation is actually the condition for existence of a derivative of Q(x).



We calculated the (instantaneous) slope of Q(x) in finding the derivative. We can then construct L(x) using the point-slope equation for this line:

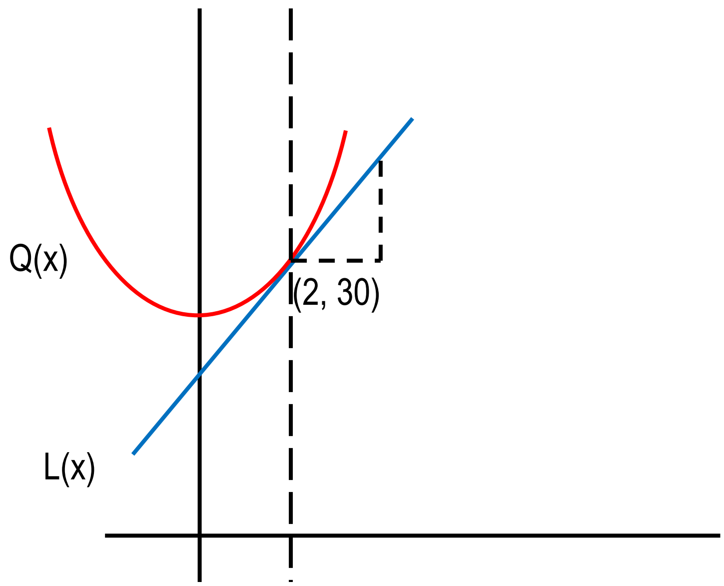
$y_2 - y_1 = m(x_2 - x_1)$

$$L(x) - 30 = Q'(2)[x - 2] = 20[x - 2]$$

$$L(x) = Q'(2)[x - 2] + 30 = 20x - 10$$

Notice that if x increases by $\Delta x = 1$ then L(x) increases by $10 \cdot \Delta x = 10$.

That means that in the triangle added at the right, ratio of the rise over the run is 10:1.



If x increases by $\Delta x = 0.001$ then L(x) increases by $10 \cdot \Delta x = 0.01$. Because L(x) approximates Q(x), we know that $\Delta Q \approx 0.01$, i.e. Q also increases by about 0.01.

$$\frac{\Delta Q}{\Delta x} \approx Q'(2) = 10$$

$$\Delta Q \approx Q'(2) \cdot \Delta x = 10 \cdot \Delta x$$

Any small change in an independent variable x is called the differential of that variable and is denoted dx. Any change in a dependent variable, like Q or y, induced by a small change in its independent variable is also called the differential of that variable dQ or dy.

Near a specific value of the input, like x = 2 here, the output differential must be calculated as a constant multiple of the input differential.

Marginal Analysis.

The derivative of the profit (or cost or revenue) function is the approximate change in profit (or cost or revenue) associated with increasing production by one unit. This change is called the ***Marginal Profit (or Cost or Revenue)***.

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Differentials and Marginal Analysis

Ex 1: Suppose we are selling 240 units per month of a product and our monthly profit as a function of monthly sales x is $P(x) = -0.0003x^4 + 17{,}500x - 1{,}000{,}000$ dollars. Use **Marginal Analysis** to find the approximate change in profit derived from increasing sales by 4 units per month. Round to the nearest cent.

Solution: $P'(x) = -0.0012x^3 + 17500$

$P'(240) = -0.0012 \cdot 240^3 + 17500 = 911.20$

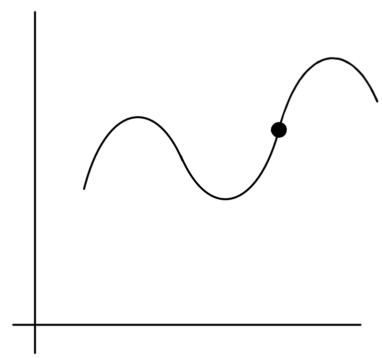
$\Delta P \approx 4 * 911.20 = 3644.8$

Notice that $P(244) - P(240) = 1967.41$

Graph this function in the window X: [0, 400]

Y: [-1000000, 3000000]

Notice that the function is near its sharpest turn when x = 240. Even though the function does not fail to be differentiable here, the derivative is less useful than it is in the straighter parts of the graph. Also, the derivative would be more useful if the change in x were a smaller percentage of 240.



More importantly, the concept of Marginal Profit arises in problems of balancing production resources used for multiple products. In these situations, the profit P(x) may be as pictured to the right. One might think that the production level yielding the absolute maximum is best for the company. However, that will deprive the company's other production systems of resources they need. The ***Point of Diminishing Returns*** is the indicated point where concavity changes from up to down (the second derivative of profit is zero). The intuition is that the company is getting the biggest return for each dollar invested in this process. Additional investment returns less and less, so those resources should be invested in other processes.

Exercises: Use marginal analysis to approximate the additional profit derived from increasing production by Δx units at the given production level with the given profit function P(x).

1. $P(x) = -0.02x^2 + 1900x - 1000000$; x = 4500; $\Delta x = 3$

2. $P(x) = -0.02x^2 + 1900x - 1000000$; x = 6000; $\Delta x = 5$

3. $P(x) = -x^3 + 30x^2 - 100x + 1000$; x = 10; $\Delta x = 2$

4. $P(x) = -x^3 + 30x^2 - 100x + 1000$; x = 15; $\Delta x = 2$

5. $P(x) = -x^3 + 30x^2 - 100x + 1000$; x = 5; $\Delta x = 2$

6. $P(x) = -0.3x^3 + 30x^2 - 100x + 1000$; x = 30; $\Delta x = 5$

7. $P(x) = -0.3x^3 + 30x^2 - 100x + 1000$; x = 15; $\Delta x = 5$

8. $P(x) = -0.3x^3 + 30x^2 - 100x + 1000$; x = 50; $\Delta x = 5$

Another Application – Variable substitution

Toward the end of the course, we look into differentials more deeply and somewhat abstractly.

In a simple case, if $u(x) = 5x + 3$ we calculate $x = \frac{u-3}{5}$. We also use the following general definition:

$$du = u' \cdot dx$$

as if we thought of $\frac{du}{dx}$ as a fraction and multiplied both sides of $\frac{du}{dx} = u'$ by dx

Ex 2: Consider the expression $3x^2\left(x^3+5\right)^7 dx$. We want to make a substitution that simplifies this expression. In calculus we want the expression to be a sum of constant multiples of power functions.

The idea is that we will try to undo a chain rule differentiation by unraveling a composition of functions. One of the composed functions will be the expression $\left(x^3+5\right)$ in parentheses. We define $u(x) = x^3 + 5$.

Then $u'(x) = 3x^2$, i.e. $\frac{du}{dx} = 3x^2$ or $du = 3x^2 \cdot dx$

Now we can rearrange the expression

$3x^2\left(x^3+5\right)^7 dx = \left(x^3+5\right)^7 3x^2 \cdot dx = u^7 \cdot du$ and it no longer involves multiplication – the factor $3x^2$ has been absorbed in the differential.

Answers:

1. 5160

2. 8300

3. 400

4. 250

5. 250

6. 4450

7. 2987.5

8. 3250